Games, graphs, and machines

Anand Deopurkar

Function or not?

Do the following rules define functions?

1. $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(s) = s^2$. ✓ not injective
2. $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(s) = s/2$. ✗
3. $f : \mathbb{Z} \to \mathbb{R}$ defined by $f(s) = s/2$. ✓
4. $f : \mathbb{R} \to \mathbb{R}$ defined by $f(s) = \sqrt{s}$. ✗. $f : \mathbb{R} \to \mathbb{C}$ Still no because $\sqrt{\ }$ is ambiguous.
5. $f : \mathbb{R} \to \mathbb{R}$ defined by ✗

$$f(s) = \begin{cases} 1 \text{ if } s > 0 \\ -1 \text{ if } s < 0 \end{cases}$$

Fix:

1. Define $f(0)$
2. Change domain to $\mathbb{R} - \{0\}$

$f: A \to B$ inj $\Rightarrow$ $|A| \leq |B|$

Not consversely.

$A = \{1, 2, 3\}$ $B = \{1, 2, 3, 4\}$

$$f: \begin{array}{l} 1 \longmapsto 1 \\ 2 \longmapsto 2 \\ 3 \longmapsto 2 \end{array}$$

$f: A \to B$

Number of functions

Suppose $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4\}$.

1. How many functions are there from A to B?
2. How many of these are injective functions?
3. How many of these are surjective functions?

all

Input	Output	
1	1 or 2 or 3 or 4	4
2	— " —	× 4
3	— " —	× 4

|Source|

|Target|

4^3

inj fun

inp	outp
1	1 or 2 or 3 or 4
2	3 possibilities
3	2 possib...

$4 \times 3 \times 2$

Number of functions (continued)

Suppose A has size n and B has size m.

How many functions are there from A to B? m^n

Suppose $n \leq m$. How many functions are there injective?

Suppose $n \geq m$. How many surjections?

e.g. $\{1,2,3,4,5\} \to \{1,2,3\}$

Answer is complicated

involves Stirling numbers.

The inverse function

Suppose $f\colon S \to T$ is a bijection.

The inverse of f is the function $g\colon T \to S$ defined by the property that if $t = f(s)$ then $s = g(t)$.

1. Suppose $f\colon \{1,2,3\} \to \{1,2,3\}$ sends $1 \mapsto 2, 2 \mapsto 3, 3 \mapsto 1$. Find its inverse g. : $1 \mapsto 3 \quad 2 \mapsto 1 \quad 3 \mapsto 2$
2. Find a function $f\colon \{1,2,3\} \to \{1,2,3\}$ which is its own inverse. $f: 1 \mapsto 1 \quad 2 \mapsto 2 \quad 3 \mapsto 3$ | $1 \mapsto 2$, $2 \mapsto 1$, $3 \mapsto 3$
3. Find a function $f\colon \mathbb{R} \to \mathbb{R}$ which is its own inverse.

Input/Output relation

Consider $R \subset \mathbb{R} \times \mathbb{R}$ defined by

$$R = \{(x, y) \mid x^3 - xy + x - 1 = 0\}.$$

Is R the input/output relation of a function $f : \mathbb{R} \to \mathbb{R}$?

On Friday